The magnitude of $\dot{s}_{irr}$ is a measure of the importance of the irreversible processes. It is given algebraically by Eq. (7) as a function of position and time. It may be evaluated for a particular fluid particle by utilizing Lagrangian coordinates. Since Eq. (7) is uncoupled from the equations of motion, it may be evaluated as an auxilliary equation during a numerical computation or after the computation is completed.

Once $\dot{s}_{irr}$ is established, Eq. (5) can be utilized for the evaluation of $s(x,t)$. In contrast to Eq. (7), a partial differential equation is to be solved for which an expression is needed for the substantial derivative. For any scalar F, the time derivative in general coordinates is related to the Cartesian time derivative by[6]

$$\frac{\partial F}{\partial t}\bigg|_{x_i} = \frac{\partial F}{\partial t}\bigg|_{\xi^j} + \frac{\partial F}{\partial \xi^k}\frac{\partial \xi^k}{\partial t} = \frac{\partial F}{\partial t} + w\cdot \mathrm{grad}F \tag{8}$$

where the velocity of the ξ^j coordinates is

$$w^k = \frac{\partial \xi^k}{\partial t}$$

With Eq. (8), the substantial derivative in Eq. (5) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\cdot \mathrm{grad}$$

where

$$u = V + w$$

is a modified velocity that represents the sum of the fluid velocity and the velocity of the ξ^j coordinates.

Appendix

The Cartesian base vectors are denoted as $\hat{I}_i$ and the covariant unitary base vectors of the general system are designated as e_j. If A is a vector and $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\pi}$ are tensors in the general system, it follows that

$$e_i\cdot e_j = g_{ij}, \quad e_i\cdot e^j = \delta_i^j$$

$$e^j = g^{ij}e_i, \quad e_i = \frac{\partial x^j}{\partial \xi^i}\hat{I}_j$$

$$J = \left|\frac{\partial x_m}{\partial \xi^n}\right| = e_i\cdot(e_j\times e_k) = g^{1/2}$$

$$\mathrm{div}A = \frac{1}{J}\frac{\partial}{\partial \xi^i}(JA^i)$$

$$\mathrm{grad}A = e^j\frac{\partial}{\partial \xi^j}(A^ie_i) = A^k_{,j}e^je_k = A^k_{,j}g^{ij}e_ie_k$$

$$(\mathrm{grad}\,A)^t = A^j_{,k}e^je_k = A^j_{,k}g^{ij}e_ie_k$$

$$A^k_{,j} = \frac{\partial A^k}{\partial \xi^j} + A^i\{{}_i{}^k{}_j\}$$

$$\{{}_i{}^k{}_j\} = \frac{\partial e_i}{\partial \xi^j}\cdot e^k$$

$$\overleftrightarrow{\sigma}:\overleftrightarrow{\pi} = \sigma_i^j\pi_j^i$$

In the above, δ_i^j is the Kronecker delta, J the Jacobian of the transformation, g the determinant of the fundamental metric tensor, and $\{{}_i{}^k{}_j\}$ the Christoffel symbol of the second kind.

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Moment Exerted on a Coning Projectile by a Spinning Liquid in a Spheroidal Cavity

Charles H. Murphy*
U.S. Army Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland

Nomenclature

a	= maximum radial distance of the liquid-filled container (for a cylinder, the radius; for a spheroid, the radial semiaxis)
c	= one-half the maximum axial dimension of the liquid-filled container (for a cylinder, one-half the length; for a spheroid, the axial semiaxis)
f_s, f_c	= fineness ratio, c/a, for a spheroid or cylinder, respectively
m_L	= mass of the liquid in the completely filled container
Re	= Reynolds number, $a^2\dot{\phi}\nu$
ϵ	= nondimensionalized damping
ν	= kinematic viscosity of the liquid
τ	= nondimensionalized frequency, $\dot{\phi}_c/\dot{\phi}$
ϕ_c	= phase angle of the coning motion
$\dot{\phi}$	= spin rate with respect to inertial axes, assumed positive

Introduction

THE prediction of the moment exerted by a spinning and coning projectile has been a problem of considerable interest to the Army for some time. For a spinning inviscid liquid in a cylindrical container, the linear liquid moment was first computed by Stewartson[1] by use of eigenfrequencies determined by the fineness ratio of the container. Wedemeyer[2]

Received Dec. 9, 1986; revision received June 29, 1987.

*Chief, Launch and Flight Division; Fellow AIAA.

introduced boundary layers on the walls of the container and was able to compute viscous corrections to Stewartson's eigenfrequencies, which could then be used in Stewartson's moment calculation. Murphy[3] then completed the linear boundary-layer theory by including all pressure and wall shear contributions to the liquid moment. The results of this improved linear theory (ILT) are expressed in terms of dimensionless liquid in-plane and liquid side-moment coefficients that can be immediately incorporated in missile angular stability calculations.

Concurrent with Wedemeyer's boundary-layer work, Greenspan[4] developed a general boundary-layer theory to compute viscous corrections to eigenfrequencies associated with a variety of containers. Kudlick[5] used this general theory to compute viscous corrections for spheroids and cylinders. Unfortunately, these calculations were marred by algebraic errors.[6] Neither Greenspan nor Kudlick showed very much interest in computing liquid moments.

Although spheroidal containers have little military interest, a spheroid is probably the only shape other than a cylinder that can be solved by analytical methods. The prediction of liquid moments in any other shaped container will have to be based on computational fluid dynamics. An analytical solution for spheroidal containers should have great value in validating general computational codes.

Motion of a rotating inviscid liquid in a spheroidal container has been studied for some time.[7,8] This inviscid theory for spheroids has been modified here by a boundary-layer correction in the same way as in the ILT derivation.[9] The resulting pressures and velocity profiles can then be used to compute liquid in-plane and liquid side-moment coefficients for various eccentricities, coning frequencies, and Reynolds numbers. In this paper, these results are discussed and compared with similar results for cylinders.

Theory

We will make the very restrictive assumption that the liquid is in steady-state response to the spinning and coning motion of the projectile. For cylindrical payloads, theoretical studies have been made to determine the effect of partially spun-up liquid[10-14] and an experimental study of the transient response to coning motion has been made.[15] These studies show that spin-up and cone-up effects are large and important to a complete understanding of the liquid payload stability problem.

The object of our linear theory is to predict the steady-state liquid moment response to coning or spiral motion. This angular motion can be described by a complex equation for the angle of attack and sideslip, α and β.

$$\beta + i\alpha = \hat{K}e^{s\phi} \tag{1}$$

where

$$\hat{K} = K_0 e^{i\phi_0}, \qquad s = (\epsilon + i)\tau$$

Although the liquid moment has components in the plane of angle of attack and perpendicular to this plane, only the perpendicular component, i.e., the liquid side moment, affects the growth of the motion and will be considered in this note. The dimensionless liquid side moment is defined as

$$C_{\mathrm{LSM}} = (\text{side moment})\ (m_L a^2 \dot{\phi}^2 K_0 \tau)^{-1} \tag{2}$$

In Ref. 3, four perturbation functions of x and r are introduced to describe the liquid pressure and the cylindrical coordinates of the liquid velocity in the earth-fixed axes. It is then shown that the perturbation functions for inviscid flow in a cylinder can be written as a series of products of trigonometric functions and Bessel functions. The perturbation pressure, e.g., has the form

$$p_s = \Sigma d_k F_k(x, r) \tag{3}$$

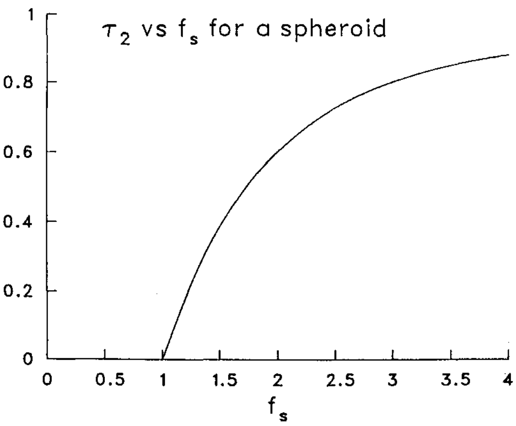

Fig. 1 τ_2 vs spheroid fineness ratio for infinite Reynolds number.

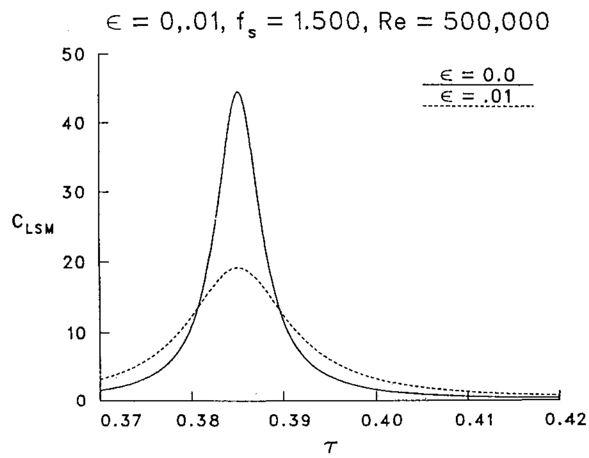

Fig. 2 C_{LSM} vs τ for $f_s = 1.5$, $R_e = 500{,}000$, and $\epsilon = 0.0$, 0.01.

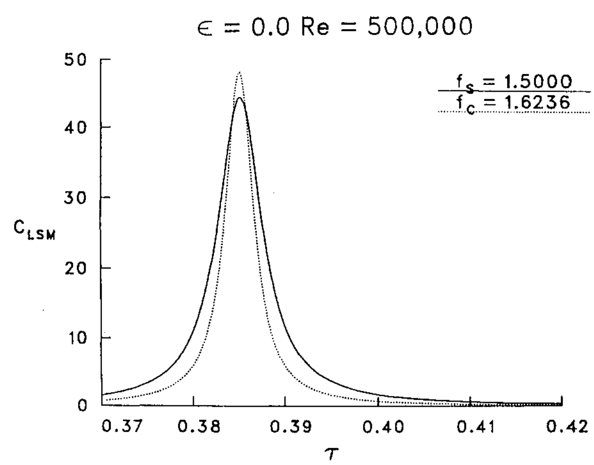

Fig. 3 Comparison of C_{LSM} vs τ for spheroid and equivalent cylinder; $f_s = 1.5$, $R_e = 500{,}000$ and $\epsilon = 0$.

where the d_k's are real constants determined by the forced coning motion to be functions of τ, ϵ, Re, and the fineness ratio of the cylinder (f_c). Unfortunately, when the d_k's are real, the corresponding side moment is zero. With the inclusion of viscosity, however, Wedemeyer's boundary-layer theory predicts d_k's with imaginary parts that vary as $Re^{-1/2}$ and the resulting side moment is not necessarily zero. The corresponding side-moment coefficient is a function of τ, ϵ, Re, and f_c. For each term in Eq. (3), there is an infinity of eigenfrequencies, τ_{kn}, for which the liquid moment has a local maximum. Fortunately, the magnitude of the maximum decays rapidly with n and only the first three values of n need be considered.

In Ref. 9, it is shown that the corresponding functions in Eq. (3) for inviscid flow in a spheroid are products of associated Legendre functions. For the forced coning motion of a spheroid, all the d_k's are zero except for d_2, which is a real function of τ, ϵ, and f_s, where f_s is the fineness ratio of the spheroid. Thus, the liquid moment is completely determined by d_2 and the function F_2. Once again, the appropriate boundary-layer analysis[9] inserts an imaginary part in d_2 and a nonzero side-moment results. F_2, however, has only one eigenvalue, τ_2, and the side moment has one maximum when τ is near τ_2. Since the liquid moment for a cylinder has a double infinity of local maxima, the situation for a spheroid is much simpler.

Discussion

The liquid side moment produced by liquid in a cylindrical container has significant local maxima at a number of eigenfrequencies. Liquid in a spheroidal container has only one significant local maximum in its liquid side moment and this is at the eigenfrequency τ_2. In Fig. 1, τ_2 is plotted as a function of f_s for f_s between 1 and 4. In flight, τ is always less than 0.4 and τ_2 is greater than this value for most of this range of fineness ratio. Thus, the primary flight stability result of this note is that low-viscosity liquids in spheroidal containers will have no flight instability if f_s is less than 1 or greater than 1.5.

Cooper[16] has coded the complete calculation for spheroid side moment for a VAX mainframe computer. In Fig. 2, C_{LSM} is plotted as a function of frequency for $f_s = 1.5$, $Re = 500{,}000$, and $\epsilon = 0$, 0.01. For zero damping, a maximum side moment of 44 occurs at $\tau_2 = 0.385$. The presence of a small amount of undamping represented by $\epsilon = 0.01$ reduces this maximum value to 19.

We can select the fineness ratio f_c of a cylindrical cavity so that its τ_1 is equal to the τ_2 of a spheroidal cavity with fineness ratio f_s. This process defines the relation between fineness ratios of spheroidal cavities and their "equivalent" cylindrical cavities.

We compare liquid side-moment coefficients in Fig. 3 for a liquid-filled spheroidal cavity with $f_s = 1.5000$ and its equivalent liquid-filled cylindrical cavity, $f_c = 1.6236$. These curves are very similar and we see that the amplitudes of the moments for the two different slopes are nearly the same when the resonance occurs at the same frequency.

Conclusions

1) Liquid moment coefficients have been computed for fully-filled spheroidal cavities as functions of fineness ratio, coning frequency, and Reynolds number.

2) Only one eigenfrequency, τ_2, is important for side-moment calculations, and it can have adverse effects on flight stability for a very limited range of fineness ratios ($1 < f_s < 1.5$).

3) The amplitudes of liquid side-moment coefficients for spheroidal cavities are very similar to those for cylindrical cavities with the same primary resonance frequency.

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Optimum Design of Structures in a Fuzzy Environment

S. S. Rao*
Purdue University, West Lafayette, Indiana

Introduction

THE theory of fuzzy sets is developed for a domain in which descriptions of activities and observations are "fuzzy," in the sense that there are no well-defined boundaries of the set of activities or observations to which the descriptions apply. The theory enables one to structure and describe activities and observations that differ from each other vaguely, to formulate them in models, and to use these models for various purposes, such as problem-solving and decision-making.

It is well known that, in practice, designers are often forced to state their design problems in precise mathematical terms rather than in terms of the real world, which may often be im-

Received March 25, 1986; revision received March 2, 1987.

*Department of Mechanical Engineering.